Spectral triples for finitely generated groups, index 0.

Sébastien Palcoux

SPECTRAL TRIPLES FOR FINITELY GENERATED GROUPS
INDEX 0

SÉBASTIEN PALCOUX

Abstract. Using Cayley graphs and Clifford algebras, we are able to give, for every finitely generated groups, a uniform construction of spectral triples with a generically non-trivial phase for the Dirac operator. Unfortunatly D_+ is index 0, but we are naturally led to an interesting classification of finitely generated groups into three types.

Warning 0.1. *This paper is just a first draft, it contains very few proofs. It is possible that some propositions are false, or that some proofs are incomplete or trivially false.*

Contents

1. Introduction

In this paper, we define even spectral triples for every finitely generated groups such that the phase of the Dirac operator is generically non-trivial but unfortunatly D_+ is index 0. We use Clifford algebra and topics in geometric group theory (see [3]) as the Cayley graph. We start giving a first construction running for a particular class of group called $\mathcal{C}$; then for the general case, we homogenize the construction. The idea is that the more we need to homogenize, the more it increases the chances that the Dirac phase is trivial. We are then led to classify the finitely generated groups into three types A_0, A_λ and A_1 according to the need of homogenization; $\mathcal{C}$ is a particular class of type A_0.

2000 *Mathematics Subject Classification.* Primary 46L87. Secondary 20F65.
Key words and phrases. non-commutative geometry; spectral triple; geometric group theory; Clifford algebra; Cayley graph; Dirac operator; finitely generated group.

2. Basic definitions

Definition 2.1. *A spectral triple* $(\mathcal{A}, H, D)$ *is given by a unital* $\star$*-algebra* $\mathcal{A}$ *representated on the Hilbert space* H*, and an unbounded operator* D*, called the Dirac operator, such that:*

(1) D *is self-adjoint.*
(2) $(D^2 + I)^{-1}$ *is compact.*
(3) $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ *is dense in* $\mathcal{A}$*.*

See the article [6] *of G. Skandalis, dedicated to A. Connes and spectral triple.*

Definition 2.2. *A group* Γ *is finitely generated if it exists a finite generating set* $S \subset \Gamma$*. We always take* S *equals to* S^{-1} *and the identity element* $e \notin S$*.*
We can also defined a group by generators and relations: $\Gamma = \langle S \mid R \rangle$*.*

3. Geometric construction

Definition 3.1. *Let* $\ell : \Gamma \to \mathbb{N}$ *be the word length related to* S*. Let* $\mathcal{G}$ *be the Cayley graph with the distance* $d(g, h) = \ell(gh^{-1})$*, see* [3]*.*

Definition 3.2. *Let* $p : \Gamma \to \mathbb{N}$ *with* $p(g)$ *the number of geodesic paths from* g *to the identity element* e*, on the Cayley graph* $\mathcal{G}$*.*

Definition 3.3. *Let* $p_s : \Gamma \to \mathbb{N}$ *with* $p_s(g)$ *the number of geodesic paths from* g *to* e*, starting by* $s \in S$*. Then* $\sum_{s \in S} p_s = p$*.*

Remark 3.4. *If* $p_s(g) \neq 0$ *then* $p_s(g) = p(gs)$*.*

Definition 3.5. *Let* $\ell^2(\Gamma)$ *be the canonical Hilbert space of base* $(e_g)_{g \in \Gamma}$ *and let* ∂_s $(s \in S)$ *be the unbounded operator definited by:*

$$\partial_s . e_g = \frac{p_s(g)}{p(g)} \ell(g) e_g .$$

∂_s *is a diagonal positive operator, so* $\partial_s^\star = \partial_s$ *and* $\partial_s \partial_{s'} = \partial_{s'} \partial_s$*.*

Example 3.6. *Let* $\Gamma = \mathbb{Z}^2$*,* $S = \{a, a^{-1}, b, b^{-1}\}$*,* $R = \{aba^{-1}b^{-1}\}$*.*
Let $g = a^n b^m = (n, m)$ *with* $n, m \in \mathbb{N}$*, then* $\ell(g) = n + m$*,* $p(g) = C^n_{n+m}$*,* $p_{a^{-1}}(g) = C^{n-1}_{n+m-1}$ *,* $p_{b^{-1}}(g) = C^{m-1}_{n+m-1}$*, and so* $\frac{C^{n-1}_{n+m-1}}{C^n_{n+m}}(n + m) = n$*.*
Let $\partial_1 = \partial_{a^{-1}}$ *and* $\partial_2 = \partial_{b^{-1}}$*, then:*

$$\partial_1 e_{(n,m)} = n e_{(n,m)} \qquad \text{and} \qquad \partial_2 e_{(n,m)} = m e_{(n,m)},$$

as for the canonical derivations !

Example 3.7. *Let* $\Gamma = \mathbb{F}_2$*,* $S = \{a, a^{-1}, b, b^{-1}\}$*,* $R = \emptyset$*. Let* $g \in \mathbb{F}_2$*,* $g \neq e$*, then* $p(g) = 1$*, let* $s \in S$ *with* $p_s(g) = 1$*, then* $\partial_s . e_g = \ell(g) . e_g$*.*

4. Clifford algebra

We quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. *Let $\Lambda(\mathbb{R}^S)$ be the exterior algebra equals to $\oplus_{k=0}^{2d}\Lambda^k(\mathbb{R}^d)$, with $2d = card(S)$ and $\Lambda^0(\mathbb{R}^d) = \mathbb{R}\Omega$. We called Ω the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.*

Definition 4.2. *Let a_v be the creation operator on $\Lambda(\mathbb{R}^S)$ defined by:*

$$a_v(v_1 \wedge ... \wedge v_r) = v \wedge v_1 \wedge ... \wedge v_r \text{ and } a_v(\Omega) = v$$

Reminder 4.3. *The dual $a_v^\star$ is called the annihilation operator, then:*

$$a_v^\star(v_1 \wedge ... \wedge v_r) = \sum_{i=0}^{r}(-1)^{i+1}(v, v_i)v_1 \wedge ...v_{i-1} \wedge v_{i+1} \wedge ... \wedge v_r \text{ and } a_v^\star(\Omega) = 0$$

Reminder 4.4. *Let $c_v = a_v + a_v^\star$, then $c_v = c_v^\star$ and $c_v c_w + c_w c_v = 2(v, w)I$.*

Definition 4.5. *The operators c_v generate the Clifford algebra Cliff$(\mathbb{R}^S)$. Note that the operators c_v are bounded and that Cliff$(\mathbb{R}^S).\Omega = \Lambda(\mathbb{R}^S)$.*

Remark 4.6. *$\mathbb{R}^S$ admits the orthonormal basis $(v_s)_{s\in S}$.*
We will write c_s instead of c_{v_s}, so that $[c_s, c_{s'}]_+ = 2\delta_{s,s'}I$.

5. Class $\mathcal{C}$ and Dirac operator

Definition 5.1. *Let H be the Hibert space $\Lambda(\mathbb{R}^S)\overline{\otimes}\ell^2(\Gamma)$, we define a self-adjoint operator D on a dense domain of H by:*

$$D = \sum_{s\in S} c_s \otimes \partial_s.$$

Lemma 5.2. *$D^2 = I \otimes (\sum_{s\in S} \partial_s^2)$.*

Proof. $D^2 = \sum_{s,s'\in S} c_s c_{s'} \otimes \partial_s \partial_{s'} = \frac{1}{2}\sum (c_s c_{s'} + c_{s'} c_s) \otimes \partial_s \partial_{s'} = I \otimes (\sum \partial_s^2)$ □

Lemma 5.3. *$(D^2 + I)^{-1}$ is compact.*

Proof. First of all $\dim(\Lambda(\mathbb{R}^S)) = 2^{2d} < \infty$.
Next $\sum_{s\in S} \frac{p_s(g)}{p(g)} = 1$, then it exists $s_o \in S$ such that $\frac{p_{s_o}(g)}{p(g)} \geq \frac{1}{2d}$.
Now $\sum_{s\in S}[\ell(g)\frac{p_s(g)}{p(g)}]^2 \geq [\ell(g)\frac{p_{s_o}(g)}{p(g)}]^2 \geq [\frac{\ell(g)}{2d}]^2$. □

Definition 5.4. *Let $\mathcal{C}$ be the class of finitely generated group Γ such that it exists a finite generating set $S \subset \Gamma$ (with $S = S^{-1}$ and $e \notin S$) such that $\forall g \in \Gamma$, $\exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h \neq e$):*

$$\left|\frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)}\right| \leq \frac{K_g}{\ell(h)}$$

Examples 5.5. *The class $\mathcal{C}$ is stable by direct or free product, it contains $\mathbb{Z}^n$, $\mathbb{F}_n$, the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see [2]).*

Warning 5.6. *From now, the group Γ is in the class $\mathcal{C}$.*

Definition 5.7. *Let $\mathcal{A} = C^\star_r\Gamma$, acting on H by $I \otimes u_g$, $g \in \Gamma$.*

Proposition 5.8. *$\{a \in \mathcal{A} \mid [D,a] \in B(H)\}$ is dense in $\mathcal{A}$.*

Proof. $[D,u_g](\Omega \otimes e_h) = ... = (\sum_{s\in S}[\ell(gh)\frac{p_s(gh)}{p(gh)} - \ell(h)\frac{p_s(h)}{p(h)}]v_s) \otimes e_{gh}$.
Now $|\ell(gh)\frac{p_s(gh)}{p(gh)} - \ell(h)\frac{p_s(h)}{p(h)}| \leq \ell(gh)|\frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)}| + |\ell(gh) - \ell(h)|\frac{p_s(h)}{p(h)}$
$\leq K_g\frac{\ell(gh)}{\ell(h)} + \ell(g) \leq K_g(\ell(g)+1) + \ell(g)$. It's bounded in h.

Now let $v \in \Lambda(\mathbb{R}^S)$ then it exists $X \in \mathrm{Cliff}(\mathbb{R}^S)$ with $v = X\Omega$.
By linearity we can restrict to $X = c_{s_1}...c_{s_r}$ with $s_i \in S$ and $i \neq j \Rightarrow s_i \neq s_j$.
Warning, the following commutant $[.,X]$ is a graded commutant:
$[D,u_g](v \otimes e_h) = ([[D,u_g],X] + (-1)^r X[D,u_g])(\Omega \otimes e_h)$
$[[D,u_g],X] = [[D,X],u_g]$
$[D,X] = [\sum_{s\in S} c_s \otimes \partial_s, X] = \sum_{s\in S}[c_s,X] \otimes \partial_s$
$[c_s,X] = 2\sum(-1)^{i+1}\delta_{s,s_i}c_{s_1}...c_{s_{i-1}}c_{s_{i+1}}...c_{s_r}$, $s_i \neq s_j$ if $i \neq j$, then:
$[D,X] = \sum X_i \otimes \partial_{s_i}$ with $X_i = (-1)^{i+1}2c_{s_1}...c_{s_{i-1}}c_{s_{i+1}}...c_{s_r}$
$[[D,X],u_g](\Omega \otimes e_h) = \sum[\ell(gh)\frac{p_{s_i}(gh)}{p(gh)} - \ell(h)\frac{p_{s_i}(h)}{p(h)}]X_i\Omega \otimes e_{gh}$.
Now X and X_i are bounded; the result follows. □

Proposition 5.9. *$D_+ : H^+ \to H-$ is a Fredholm operator of index 0.*

Proposition 5.10. *For $t > 0$, the operator $e^{-t\mathcal{D}^2}$ is trass-class.*

Theorem 5.11. *$(\mathcal{A}, H, D)$ is an even θ-summable spectral triple, the Dirac phase is generically non trivial, but D_+ is index 0.*

Proof. We use lemma 5.3 and proposition 5.8. The phase is generically non-trivial, because it's non-trivial for $\Gamma = \mathbb{Z}^2$ or $\mathbb{F}_2$. □

6. Homogenization and general case

The class $\mathcal{C}$ doesn't contain every finitely generated group, we have the following non-automatic (see [2]) counterexample:

Counterexample 6.1. *Baumslag-Solitar group $B(2,1)$, $S = \{a, a^{-1}, b, b^{-1}\}$, $R = \{a^2ba^{-1}b^{-1}\}$, $a^{2^n} = b^{n-1}a^2b^{1-n}$, $\ell(a^{2^n}) = 2n$ with $n > 1$,*

$$|\frac{p_b(a.a^{2^n})}{p(a.a^{2^n})} - \frac{p_b(a^{2^n})}{p(a^{2^n})}| = |\frac{2}{4} - \frac{2}{2}| = \frac{1}{2}$$

It should rest to prove that such a failure is independant of the presentation...

Then, to obtain a construction in the general case, we need to operate a little homogenization. Let $\Gamma = \langle S \mid R\rangle$ be a finitely generated group.

Definition 6.2. *Let $\mathbb{B}^n_\Gamma = \{g \in \Gamma \mid \ell(g) \leq n\}$ the ball of radius n.*

Definition 6.3. *Let $\mathbb{S}^n_\Gamma = \{g \in \Gamma \mid \ell(g) = n\}$ the sphere of radius n.*

Definition 6.4. *Let μ be the probability measure on Γ defined by:*

$$\mu(g) = (\#(\mathbb{S}_\Gamma^{\ell(g)})2^{\ell(g)+1})^{-1}.$$

Definition 6.5. *Let $E_{\Gamma,S}$ be the set of smooth functions $f : \mathbb{R}_+ \to \mathbb{R}$ with:*
(1) *$f(0) = 0$, $f \geq 0$, $f' \geq 0$ and $f'' \leq 0$.*
(2) *$\lambda_f := lim_{x\to\infty}(f'(x)) \leq 1$.*
(3) *$\forall g \in \Gamma$, $\exists K_g \in \mathbb{R}_+$ such that $\forall s \in S$ and $\forall h \in \Gamma$ (with $h \neq e$):*

$$|\sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \frac{\mu(\gamma gh)}{\mu(\mathbb{B}_\Gamma^{f(\ell(gh))}.gh)} - \sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h)}| \leq \frac{K_g}{\ell(h)}$$

Remark 6.6. *The condition (2) is well-defined because f' is decreasing and minorated by 0, so convergent at infinity.*

Remark 6.7. *Let $f \in E_{\Gamma,S}$, if $\exists\alpha > 0$ with $f(\alpha) = 0$ then $f = 0$.*

Lemma 6.8. *$E_{\Gamma,S}$ is non empty.*

Proof. We will show that $f(x) = x + log_2(x + 1)$ defines a fonction in $E_{\Gamma,S}$. First of all $\mathbb{B}_\Gamma^{log_2(\ell(g))} \subset (\mathbb{B}_\Gamma^{f(\ell(g))}.g)$, then $\mu(\mathbb{B}_\Gamma^{f(\ell(g))}.g) \geq \mu(\mathbb{B}_\Gamma^{log_2(\ell(g))}) \geq 1 - \frac{1}{\ell(g)}$.

$$|\sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \frac{\mu(\gamma gh)}{\mu(\mathbb{B}_\Gamma^{f(\ell(gh))}.gh)} - \sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h)}| \leq$$

$$|(1/\mu(\mathbb{B}_\Gamma^{f(\ell(gh))}.gh) - 1) \sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)}\mu(\gamma gh)|+$$

$$|(1/\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h) - 1) \sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h)}|$$

$$+|\sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)}\mu(\gamma gh) - \sum_{\gamma\in\mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)}\mu(\gamma h)| \leq$$

$$1/\ell(gh) + 1/\ell(h) + |\sum_{\gamma\in\Gamma\backslash\mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)}\mu(\gamma gh) - \sum_{\gamma\in\Gamma\backslash\mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)}\mu(\gamma h)|+$$

$$|\sum_{\gamma\in\Gamma} \frac{p_s(\gamma gh)}{p(\gamma gh)}\mu(\gamma gh) - \sum_{\gamma\in\Gamma} \frac{p_s(\gamma h)}{p(\gamma h)}\mu(\gamma h)|$$

$$\leq 2/\ell(gh) + 2/\ell(h) \leq 2/(\ell(g)\ell(h) + \ell(h)) + 2/\ell(h) = \frac{1/(\ell(g)+1)}{\ell(h)}$$

The others conditions for f to be in $E_{\Gamma,S}$ are evidents. □

Remark 6.9. *The class $\mathcal{C}$ is the class of group Γ for which $\exists S$ with $0 \in E_{\Gamma,S}$.*

We are led to a classification of finitely generated groups into three types:

Definition 6.10. *Let* $\lambda_\Gamma := min_{S\subset\Gamma} min_{f\in E_{\Gamma,S}} \lambda_f$*, with* S *and* λ_f *as previouly. By definition* $\lambda_\Gamma \in [0,1]$ *then:*

$\mathbf{A_0}$: Γ *is a type* A_0 *group if* $\lambda_\Gamma = 0$ *and* $\exists f$ *with* $\lambda_f = 0$.
$\mathbf{A}_\lambda$: Γ *is a type* A_λ *group if it is not* A_0 *and* $0 \leq \lambda_\Gamma < 1$ *.*
$\mathbf{A_1}$: Γ *is a type* A_1 *group if* $\lambda_\Gamma = 1$ *.*

Definition 6.11. *Let* A_0^+ *be the subclass of type* A_λ *groups with* $\lambda_\Gamma = 0$.

Remark 6.12. *Every groups of the class* $\mathcal{C}$ *(as* $\mathbb{Z}^n$, $\mathbb{F}^n$ *etc...) are of type* A_0.

Conjecture 6.13. *Every Baumslag-Solitar groups* $B(n,m)$ *are of type* A_0.

Problem 6.14. *Existence of type* A_0^+, A_λ *or* A_1 *groups.*

Definition 6.15. *Let* $\Gamma = \langle S \mid R\rangle$ *be a finitely generated group, let* $f \in E_{\Gamma,S}$ *be a function with (almost) minimal growth, we define* $\widetilde{D}$ *as* D *using:*

$$\widetilde{\partial}_s.e_g = [\sum_{\gamma\in\mathbb{B}^\Gamma_{f(\ell(g))}} \frac{p_s(\gamma g)}{p(\gamma g)} \frac{\mu(\gamma g)}{\mu(\mathbb{B}_\Gamma^{f(\ell(g))}.g)}]\ell(g)e_g.$$

Remark 6.16. *For the class* $\mathcal{C}$*, we can take* $f = 0$ *and so* $\widetilde{D} = D$.

Theorem 6.17. $(\mathcal{A}, H, \widetilde{D})$ *is an even* θ*-summable spectral triple, the Dirac phase is generically non trivial but* $\widetilde{D}_+$ *is index* 0.

Proof. The proof runs exactly as for theorem 5.11. □

Definition 6.18. *Let* $\mathcal{C}^\perp$ *be the class of groups* Γ *such that* $\forall S$, $\forall f \in E_{\Gamma,S}$*, the phase of* $\widetilde{D}$ *is always trivial.*

Problem 6.19. *Existence of groups of class* $\mathcal{C}^\perp$.

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Institut de Mathématiques de Luminy, Marseille, France.
E-mail address: palcoux@iml.univ-mrs.fr, http://iml.univ-mrs.fr/∼palcoux